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Tetrad equations for the two-component neutrino field in general relativity

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Abstract. Weyl's neutrino equations are exhibited as equivalent tetrad equations, and the tetrad conditions for a pure radiation neutrino gravitational field are established. It is shown that the neutrino flux vector for the radiation field defines a shear-free and twist-free family of geodesics. A class of metrics is found which can be interpreted either as neutrino radiation or as null Einstein–Maxwell fields.

1. Introduction

In this paper we are using two-component spinors in the notation of Newman and Penrose (1962) with space–time signature -2, the spinor indices being raised and lowered by

$$\xi^A = \epsilon^{AB}\xi_B, \qquad \xi_A = \xi^B\epsilon_{BA}.$$

In this notation Weyl's equations for the two-component neutrino are

$$\sigma^{\alpha\dot{A}B}\xi_{B|\alpha} = 0, \qquad \sigma^{\alpha\dot{A}B}\xi_{\dot{A}|\alpha} = 0 \tag{1.1}$$

where $\xi_{B|\alpha}$ denotes the spinor derivative defined by Newman and Penrose. The fundamental metric spinor $\epsilon_{AB}, \epsilon^{AB}$ and the generalized Pauli matrices $\sigma^{\mu\dot{A}B}$ are constants with respect to spinor differentiation, and the Pauli matrices have the properties

$$\sigma_\mu{}^{\dot{A}B}\sigma_{\nu B\dot{C}} + \sigma_\nu{}^{\dot{A}B}\sigma_{\mu B\dot{C}} = g_{\mu\nu}\delta^{\dot{A}}{}_{\dot{C}}$$
$$\sigma^{\alpha\dot{A}B}\sigma_{\alpha\dot{C}D} = \delta^{\dot{A}}{}_{\dot{C}}\delta^B{}_D.$$

Given ξ^A we can define a second basis spinor χ^A such that

$$\xi_A\chi^B - \xi^B\chi_A = \delta_A{}^B. \tag{1.2}$$

Using these we can construct a tetrad of null vectors

$$l_\mu = \xi^{\dot{A}}\sigma_{\mu\dot{A}B}\xi^B, \qquad n_\mu = \chi^{\dot{A}}\sigma_{\mu\dot{A}B}\chi^B$$
$$m_\mu = \xi^{\dot{A}}\sigma_{\mu\dot{A}B}\chi^B \qquad \bar{m}_\mu = \xi^A\sigma_{\mu A\dot{B}}\chi^{\dot{B}}$$

where $\bar{m}_\mu$ is the complex conjugate of m_μ, which have the properties

$$l_\alpha n^\alpha = 1, \qquad l_\alpha m^\alpha = 0, \qquad n_\alpha m^\alpha = 0, \qquad m_\alpha\bar{m}^\alpha = -1$$
$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu\bar{m}_\nu - \bar{m}_\mu m_\nu$$

l_μ and n_μ being real null vectors. We can also construct the spin tensor

$$S_{\mu\nu} = \xi_{\dot{A}}\sigma_\mu{}^{\dot{A}B}\sigma_{\nu B\dot{C}}\xi^{\dot{C}}$$

which is self dual, i.e.

$$S_{\mu\nu} = \tfrac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma}.$$

2. Tetrad equations

If the spinor ξ^A is a solution of Weyl's equation (1.1), the null vector l_μ is the neutrino flux vector and is such that $l^\nu{}_{;\nu} = 0$, the semi-colon denoting tensor covariant differentiation. It will follow from the identity

$$S_\mu{}^\nu{}_{|\nu} = 2\xi_{\dot{A}}\sigma_\mu{}^{\dot{A}B}\sigma^\nu{}_{B\dot{C}}\xi^{\dot{C}}{}_{|\nu} + \xi^{\dot{A}}\xi_{\dot{A}|\mu}$$

that Weyl's equations are exactly equivalent to the tensor equation (Penney 1965, Griffiths and Newing 1970)

$$S_\mu{}^\nu{}_{;\nu} = H_\mu \tag{2.1}$$

where $H_\mu = \xi^{\dot{A}}\xi_{\dot{A}|\mu}$. For (1.1) obviously implies (2.1) and (2.1) implies that

$$\begin{aligned} 0 &= \sigma^\mu{}_{\dot{D}E}(S_\mu{}^\nu{}_{|\nu} - H_\mu) \\ &= \sigma_{\mu\dot{D}E}\sigma^{\mu\dot{A}B}\sigma^\nu{}_{B\dot{C}}\xi_{\dot{A}}\xi^{\dot{C}}{}_{|\nu} \\ &= \delta_{\dot{D}}{}^{\dot{A}}\delta_E{}^B\sigma^\nu{}_{B\dot{C}}\xi_{\dot{A}}\xi^{\dot{C}}{}_{|\nu} \\ &= \xi_{\dot{D}}\sigma^\nu{}_{E\dot{C}}\xi^{\dot{C}}{}_{|\nu} \end{aligned}$$

that is,

$$\sigma^\nu{}_{E\dot{C}}\xi^{\dot{C}}{}_{|\nu} = 0.$$

With the help of (1.2) it may be shown that $S_{\mu\nu} = l_\mu m_\nu - m_\mu l_\nu$ and $H_\mu = m^\alpha l_{\alpha;\mu}$, and the tensor equation (2.1) may be expressed in the tetrad form

$$(l_\mu m^\nu - m_\mu l^\nu)_{;\nu} = m^\alpha l_{\alpha;\mu}. \tag{2.2}$$

With suitable units the energy–momentum tensor for the neutrino field will be taken to be (Bergmann 1960)

$$E_{\mu\nu} = \mathrm{i}(\xi^A{}_{|\nu}\sigma_{\mu A\dot{B}}\xi^{\dot{B}} + \xi^A{}_{|\mu}\sigma_{\nu A\dot{B}}\xi^{\dot{B}} - \xi^A\sigma_{\mu A\dot{B}}\xi^{\dot{B}}{}_{|\nu} - \xi^A\sigma_{\nu A\dot{B}}\xi^{\dot{B}}{}_{|\mu})$$

and again using (1.2) this may be expressed in the tetrad form

$$E_{\mu\nu} = \mathrm{i}(H_\mu\bar{m}_\nu + H_\nu\bar{m}_\mu - \bar{H}_\mu m_\nu - \bar{H}_\nu m_\mu + P_\mu l_\nu + P_\nu l_\mu) \tag{2.3}$$

where

$$P_\mu = \xi_{A|\mu}\chi^A - \xi_{\dot{A}|\mu}\chi^{\dot{A}} = \bar{m}^\alpha m_{\alpha;\mu}.$$

Weyl's equations imply that the trace of the energy–momentum tensor is zero and so, in suitable units, the gravitational equations for the combined neutrino gravitational field will be taken to be

$$R_{\mu\nu} + E_{\mu\nu} = 0. \tag{2.4}$$

A given space–time will admit a neutrino field if a null tetrad can be constructed satisfying equations (2.2) and (2.4) with $E_{\mu\nu}$ given by (2.3).

3. The pure radiation field

If the tetrad vector l_μ is interpreted as a neutrino flux vector, it is reasonable to take the energy–momentum tensor of a neutrino pure radiation field to be

$$E_{\mu\nu} = \lambda^2 l_\mu l_\nu. \tag{3.1}$$

Then since $l^\alpha E_{\alpha\nu} = 0$ it follows from (2.3) that

$$l^\alpha P_\alpha = 0$$

and

$$l^\alpha H_\alpha = 0.$$

Similarly from $m^\alpha E_{\alpha\nu} = 0$ we obtain the restriction that H_μ is of the form

$$H_\mu = al_\mu + bm_\mu \tag{3.2}$$

where $a = m^\alpha P_\alpha$ and b is real. These conditions imply that P_μ is of the form

$$P_\mu = \mathrm{i}Al_\mu + \bar{a}m_\mu - a\bar{m}_\mu \tag{3.3}$$

where A is real. If we substitute the expressions (3.2) and (3.3) into (2.3), $E_{\mu\nu}$ reduces to the form (3.1) if $A = -\frac{1}{2}\lambda^2$.

For a radiation field of this type the condition that $E^{\mu\nu}{}_{;\nu} = 0$ implies that

$$\frac{2\lambda_{,\nu}}{\lambda} l^\nu l^\mu + l^\mu{}_{;\nu} l^\nu = 0$$

since $l^\nu{}_{;\nu} = 0$. This then gives the condition that

$$l^\nu l_{\alpha;\nu} l_\beta = l^\nu l_{\beta;\nu} l_\alpha.$$

We also have for the pure radiation field that

$$m^\nu m^\alpha l_{\alpha;\nu} = 0.$$

These two conditions (Sachs 1961, Robinson 1961) imply that for a neutrino pure radiation field the flux vector l_μ defines a shear-free family of geodesics.

Now the neutrino equation (2.2) for this radiation field gives the condition $\bar{m}^\alpha H_\alpha - m^\alpha \bar{H}_\alpha = 0$ which becomes

$$(\bar{m}^\alpha m^\beta - m^\alpha \bar{m}^\beta) l_{\alpha;\beta} = 0$$

i.e. (Goodinson 1969)

$$\epsilon^{\kappa\lambda\mu\nu} l_\lambda l_{\mu;\nu} = 0.$$

For this particular case, therefore, the neutrino flux vector l_μ satisfies the same condition as the propagation vector for a null electromagnetic field corresponding to Peres' exceptional case (Peres 1961, Geroch 1966). For this condition the family of geodesics is also twist-free.

These results may be summarized in the theorem—

Theorem: If in a given space–time the vectors l_μ and m_μ define a neutrino field, the field is a pure radiation field of the type $E_{\mu\nu} = \lambda^2 l_\mu l_\nu$ if, and only if,

$$m^\alpha l_{\alpha;\mu} = al_\mu + bm_\mu$$
$$\bar{m}^\alpha m_{\alpha;\mu} = \mathrm{i}Al_\mu + \bar{a}m_\mu - a\bar{m}_\mu$$

where the coefficients b and A are real. In this case the flux vector l_μ defines a shear-free and twist-free family of geodesics, and is a scalar multiple of a gradiant.

4. A metric admitting both neutrino and null electromagnetic fields

Geroch (1966) has shown that space times defined by a metric tensor of the form

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & h_1 & h_2 & h_3 \\ 0 & h_2 & -1 & 0 \\ 0 & h_3 & 0 & -1 \end{pmatrix}$$

where the coefficients h_n are independent of the coordinate x^0, admit null electromagnetic fields for which the propagation vector L_μ satisfies the condition

$$\epsilon^{\kappa\lambda\mu\nu}L_\lambda L_{\mu;\nu} = 0.$$

In this case the electromagnetic field is not determined uniquely by the metric.

Introducing the null tetrad

$$L_\mu = e^\rho \delta_\mu{}^1, \qquad N_\mu = e^{-\rho}\{\delta_\mu{}^0 + \tfrac{1}{2}(h_1 + h_2{}^2 + h_3{}^2)\delta_\mu{}^1\}$$

$$M_\mu = \frac{1}{\sqrt{2}}\{(h_2 - ih_3)\delta_\mu{}^1 - \delta_\mu{}^2 + i\delta_\mu{}^3\}$$

the electromagnetic field is defined by the self dual tensor (Goodinson and Newing 1969)

$$\omega_{\mu\nu} = e^{i\psi}(L_\mu M_\nu - M_\mu L_\nu)$$

and the vacuum field equations $\omega^{\mu\nu}{}_{;\nu} = 0$ imply that

$$\psi_{,2} = \rho_{,3}, \qquad \psi_{,3} = -\rho_{,2}, \qquad \psi_{,0} = 0, \qquad \rho_{,0} = 0$$

and hence that $\rho_{,22} + \rho_{,33} = 0$.

The Ricci tensor for the metric is

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -k^2 & \tfrac{1}{2}\gamma_{,3} & -\tfrac{1}{2}\gamma_{,2} \\ 0 & \tfrac{1}{2}\gamma_3 & 0 & 0 \\ 0 & -\tfrac{1}{2}\gamma_{,2} & 0 & 0 \end{pmatrix}$$

where

$$\gamma = h_{3,2} - h_{2,3}$$

and

$$-k^2 = h_{2,21} + h_{3,31} - \tfrac{1}{2}(h_{1,22} + h_{1,33}) - \tfrac{1}{2}\gamma^2$$

and in suitable units the gravitational field equations are

$$R_{\mu\nu} = -L_\mu L_\nu$$

so that γ must be a function of x^1 only and $e^\rho = k$. Thus $\ln k$ must be a solution of the two-dimensional Laplace equation

$$(\ln k)_{,22} + (\ln k)_{,33} = 0.$$

Now consider the possibility of finding a solution to the neutrino gravitational field equations for the same metric. First we notice that the tetrad equations for a neutrino gravitational field are invariant with respect to ψ transformations (Peres 1961) but are dependent upon the parameter ϕ of the ϕ transformation. So we can consider defining a neutrino field with the vectors

$$l_\mu = e^\theta \delta_\mu{}^1, \qquad m_\mu = e^{i\phi}M_\mu.$$

For the given metric, $H_\mu = 0$ and the neutrino equation (2.2) implies that

$$\phi_{,2} = \theta_{,3}, \qquad \phi_{,3} = -\theta_{,2}, \qquad \phi_{,0} = 0, \qquad \theta_{,0} = 0.$$

The vector $P_\nu = \bar{m}^\alpha m_{\alpha|\nu}$ is given by

$$P_\nu = \tfrac{1}{2}\, i\gamma\delta_\nu{}^1 - i\phi_{,\nu}.$$

The conditions for a pure radiation field imply that ϕ, and hence θ, is a function of x^1 only, and the energy–momentum tensor then becomes

$$E_{\mu\nu} = (2\phi_{,1} - \gamma)\, e^{\theta} \delta_{\mu}{}^{1} \delta_{\nu}{}^{1}.$$

The metric can therefore also be interpreted as admitting a neutrino gravitational field if

$$k^2 = (2\phi_{,1} - \gamma)\, e^{\theta} \tag{4.1}$$

where γ, θ and ϕ are functions of x^1 only. This solution is therefore a particular case of the above solution for an electromagnetic field. So the metric can be interpreted as admitting either a null electromagnetic field or a neutrino pure radiation field. As in the electromagnetic case, the neutrino field is not determined uniquely by the metric, since the two neutrino parameters $\theta(x^1)$ and $\phi(x^1)$ are subject to the single restriction (4.1).

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